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**ANALYSIS OF KALMAN FILTER MECHANIZATIONS
FOR AN IDEALIZED ROCKET LAUNCH**

by
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ABSTRACT

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1. Introduction

A large class of estimation problems is concerned with finding an optimal estimate of some quantity (an unknown parameter, a random variable, or a random signal) when a linear function of this quantity, corrupted by an additive noise, is available for generating the estimate. However, the class of estimation problems most often encountered are those in which the unknown quantity is describable by equations which are not linear functions. Then, the theory of Kalman [1] and Kalman and Bucy [2], developed to indicate optimal state estimates of linear systems, no longer yields an optimal estimate. Extensions or new theory work to include a large class of truly nonlinear systems generally has not been successfully developed and is the topic of current research. A partial solution, allowing application of their theory in many instances to practical nonlinear systems, is based on work by Kushner [3], Cox [4], Bucy [5], and others.

The fundamental assumption [6] is that a nominal solution of the system's nonlinear differential equations must exist. This solution must provide a "good" approximation to the actual behavior of the system. It is "good" if the difference between the nominal and actual solution can be described by a set of linear differential equations (sometimes called linear perturbation equations). In practice the nominal equations or nominal trajectory may not be available a priori. For example, the navigation of a terrestrial vehicle may not be predetermined in the sense that the trajectory of a ballistic missile or space booster is predetermined. Without a nominal trajectory, the linear perturbation equations may be obtained as the difference between the actual solution and the current estimate of the nominal trajectory. This process, called relinearization by Bryson and Ho [7], may prove to be even more accurate in estimating the state variable than when linearizing about the nominal.

2. Purpose

It is the purpose of this study to evaluate two mechanizations of a pseudo nonlinear Kalman-Bucy filter. Performance will be evaluated of a filter that estimates the states of a nonlinear system which has been linearized about the nominal, called a linearized Kalman filter [8], versus one that has been relinearized about the current estimate, called an extended Kalman filter [8].

3. Problem

The problem is to estimate various parameters for a vehicle launched from an earth assumed to be spherical, airless, and nonrotating. The launch point and trajectory of the vehicle are coplanar with a tracking

station located downrange from the launch site. The tracking radar measures slant range and elevation of the vehicle and yields an output which is corrupted by noise. The noisy output is processed to give estimates of the vehicle's states which describe its trajectory. For this study the trajectory is assumed simply to be generated by a gravity turn. After launch the rocket is disturbed in the cross track direction by a random specific force which for realism could be considered a gusting wind disturbance.

4. Model of the System

All of the equations for the launch problem can be inferred from the geometry presented in Figure 1.

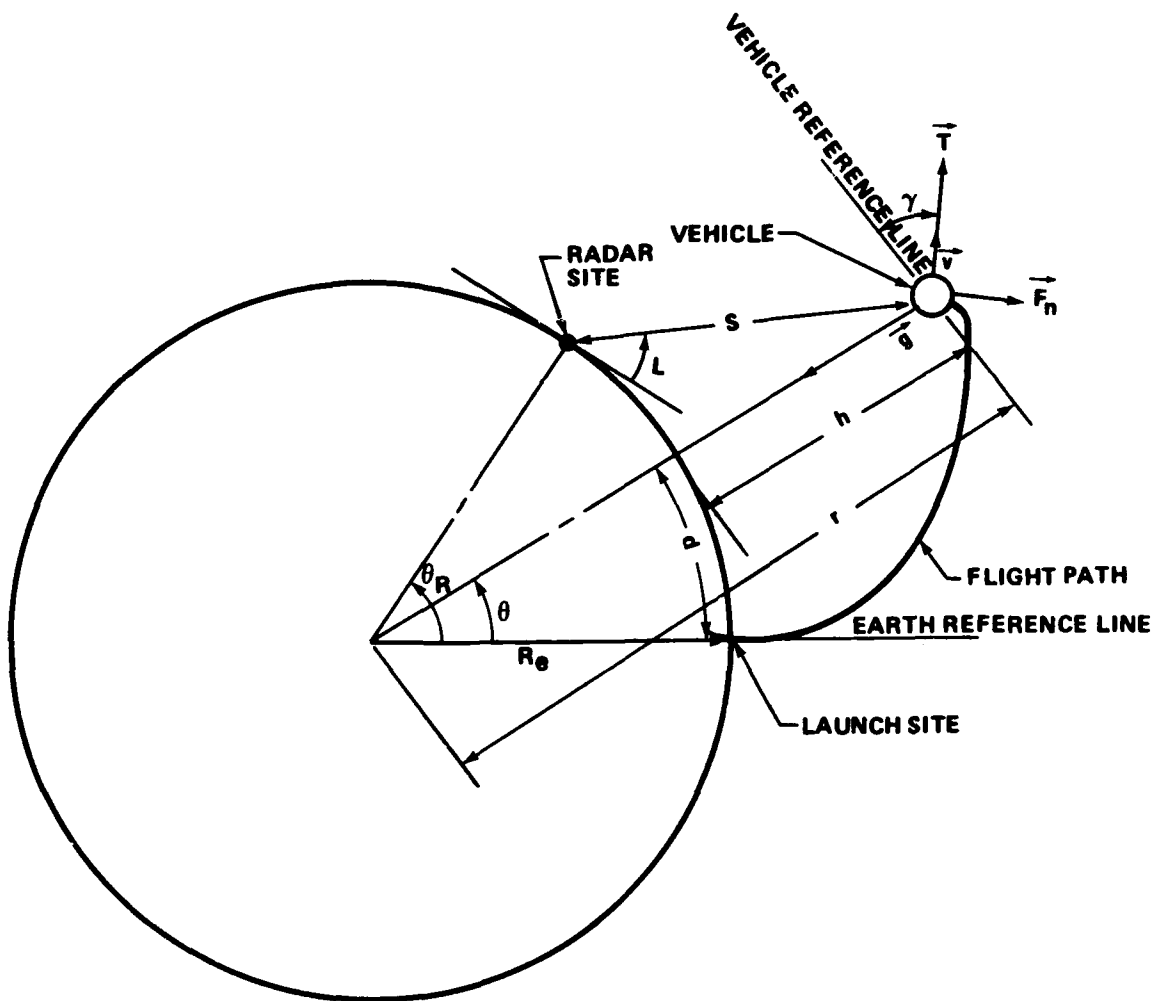


Figure 1. Geometry of the Problem

The following symbols are applied in Figure 1:

$R_e \triangleq$ earth's radius

$L \triangleq$ elevation angle between radar site and vehicle

$\theta \triangleq$ reference angle between launch site and vehicle

$\theta_R \triangleq$ reference angle between launch site and radar site

$h \triangleq$ altitude above the earth's surface

$S \triangleq$ slant range distance from radar site to vehicle

$\vec{g} \triangleq$ earth's gravity vector with magnitude at zero altitude

$\vec{v} \triangleq$ velocity of the vehicle with respect to an inertial basis

$\vec{F}_n \triangleq$ disturbing force orthogonal to \vec{v}

$\vec{T} \triangleq$ vehicles thrust vector with respect to an inertial basis

$\gamma \triangleq$ flight path angle between velocity vector and a vehicle fixed reference line

$d \triangleq$ distance of the vehicle downrange from the launch site

$r \triangleq h + R_e$.

The following state variable equations defining the vehicle's trajectory are obtainable from the given geometry of Figure 1:

$$\dot{h} = v \sin \gamma \quad , \quad (1)$$

$$\dot{d} = \frac{R_e v \cos \gamma}{h + R_e} \quad , \quad (2)$$

$$\dot{v} = \frac{-k \sin \gamma}{(h + R_e)^2} + g \left(\frac{T}{w_o} \right) \frac{1}{1 - \left(\frac{T}{w_o} \right) \frac{t}{I_{sp}}} \quad , \quad (3)$$

$$\dot{\gamma} = \frac{d}{R_e} - \frac{k \cos \gamma}{v(h + R_e)^2} + \frac{F_n}{v} \quad , \quad (4)$$

where

$$k \triangleq gR_e^2$$

$t \triangleq$ time of flight measured in seconds from the launch

$I_{sp} \triangleq$ specific impulse of the rocket fuel

$w_0 \triangleq$ initial weight of the total vehicle.

The slant range and elevation equations as a function of the states are obtained from simple trigonometric identities. From the Law of Cosines:

$$S^2 = r^2 + R_e^2 - 2 R_e r \cos (\theta_R - \theta) \quad , \quad (5)$$

so that

$$S = \sqrt{r^2 + R_e^2 - 2 R_e r \cos (\theta_R - \theta)} \quad . \quad (6)$$

From the Law of Sines:

$$\frac{S}{\sin (\theta_R - \theta)} = \frac{r}{\sin \left(\frac{\pi}{2} + L \right)} = \frac{R_e}{\sin \left(\pi - \left(\frac{\pi}{2} + L \right) - (\theta_R - \theta) \right)} \quad . \quad (7)$$

Expanding the middle term of Equation (7),

$$\begin{aligned} \sin \left(\frac{\pi}{2} + L \right) &= \sin \frac{\pi}{2} \cos L + \cos \frac{\pi}{2} \sin L \\ &= \cos L \quad . \end{aligned} \quad (8)$$

Thus,

$$\frac{r}{\cos L} = \frac{S}{\sin (\theta_R - \theta)} \quad , \quad (9)$$

and

$$L = \arccos \left(\frac{r \sin(\theta_R - \theta)}{\sqrt{r^2 + R_e^2 - 2 R_e r \cos(\theta_R - \theta)}} \right) . \quad (10)$$

Alternatively, if the right equality of Equation (7) is used the elevation angle is given by

$$L = \arctan \left[\frac{\cos(\theta_R - \theta) - \frac{R_e}{r}}{\sin(\theta_R - \theta)} \right] . \quad (11)$$

5. Approximately Nonlinear Kalman-Bucy Filter Mechanizations

The filter equations developed by Kalman and Bucy were derived under the assumption that the system disturbances and the measurement errors were random variables described by Gaussian statistics, and that the plant was describable by linear equations. The resulting filter then gave the optimal estimates of the states.

The filter equations in this study are modeled as a continuous nonlinear system. That is, unlike a discrete model, the transition of each state from one increment of time to another is considered to be a smooth continuous process. Furthermore, the observations measured with the radar tracking system are also continuous. Thus the system nonlinear dynamics are given by

$$\dot{x} = f[x(t), w(t), t] , \quad (12)$$

where

$x \triangleq$ n-vector of states

$w \triangleq$ m-vector of process noise

$t \triangleq$ time.

The observation equation is also nonlinear and given by

$$z(t) = h[x(t), t] + v(t) \quad (13)$$

where

$z \triangleq$ p-vector of observations

$v \triangleq$ k-vector of measurement noise.

The a priori statistics are given by zero means; i.e.,

$$E\{w(t)\} = 0 \quad , \quad (14)$$

$$E\{v(t)\} = 0 \quad , \quad (15)$$

where $E\{\cdot\}$ is used to denote the expected value of a quantity. The noise covariance matrices are

$$E\{w(t), w^T(\tau)\} = Q \delta(t - \tau) \quad , \quad (16)$$

$$E\{v(t), v^T(\tau)\} = R \delta(t - \tau) \quad , \quad (17)$$

$$E\{w(t), v(\tau)\} = 0 \quad , \quad (18)$$

where

$\delta(t - \tau) \triangleq$ Dirac delta function.

To estimate the states a suboptimal nonlinear filter is used which estimates their deviation from the nominal trajectory or the current estimated trajectory. The nominal trajectory is given by the nonlinear state equations with the white process noise ignored, i.e.,

$$\dot{x}_N = f[x_N(t), t] \quad , \quad (19)$$

where

$x_N \triangleq$ n-vector of nominal states.

The actual trajectory is the trajectory represented by the launch and boost itself. In this study it is simulated by Equation (12) which contains the process noise as would be expected in a realistic situation. To linearize about the nominal state, define a small error as the difference between the nominal state and the actual state

$$\delta x(t) \triangleq \tilde{x}(t) = x(t) - x_N(t) \quad . \quad (20)$$

It is apparent that

$$\delta \dot{x}(t) = \dot{x}(t) - \dot{x}_N(t) \quad . \quad (21)$$

Putting Equation (21) into Equation (12) yields

$$\dot{x}(t) = \dot{x}_N(t) + \delta \dot{x}(t) = f[x_N(t) + \delta x(t), w(t) + \delta w(t), t] \quad . \quad (22)$$

Expanding the right side of Equation (22) via a Taylor series,

$$\begin{aligned} f[x_N(t) + \delta x(t), w(t) + \delta w(t), t] = & f[x_N(t), w(t), t] + \left. \frac{\partial f[x_N(t), w(t), t]}{\partial x_N(t)} \right|_{x_N} \delta x(t) \\ & + \left. \frac{\partial f[x_N(t), w(t), t]}{\partial w(t)} \right|_{w_N} \delta w(t) + 0^{2+} \quad , \end{aligned} \quad (23)$$

where

$$\left. \frac{\partial f[x_N(t), w(t), t]}{\partial x_N(t)} \right|_{x_N} \triangleq \text{evaluation of the partial derivative at the nominal state}$$

$$\left. \frac{\partial f[x_N(t), w(t), t]}{\partial w(t)} \right|_{w_N} \triangleq \text{evaluation of the partial derivative about the conditional mean}$$

$0^{2+} \triangleq$ partial derivative terms of order two and higher.

Thus a differential equation that gives the deviation between the actual state and the nominal state to first order is

$$\delta \dot{x} = \left. \frac{\partial f[x_N(t), w(t), t]}{\partial x_N(t)} \right|_{x_N} \delta x(t) + \left. \frac{\partial f[x_N(t), w(t), t]}{\partial w(t)} \right|_{w_N} \delta w(t) \quad . \quad (24)$$

(state deviation linearized about the nominal)

In a manner which uses the Taylor series expansion as above, a differential equation can be formulated that describes the state evaluated about the estimate. By defining the error as the difference

$$x(t) \triangleq \tilde{x}(t) = x(t) - \hat{x}(t) \quad , \quad (25)$$

so that

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t) \quad (26)$$

and proceeding as before, the result is

$$\dot{\tilde{x}} = \left. \frac{\partial f[\hat{x}(t), w(t), t]}{\partial \hat{x}(t)} \right|_{\hat{x}} x(t) + \left. \frac{\partial f[\hat{x}(t), w(t), t]}{\partial w(t)} \right|_{\hat{w}} w(t) \quad , \quad (27)$$

(state linearized about the estimate)

where the partial derivative for the noise is evaluated about the expected value. This is the conditional mean, as before.

The observations given by Equation (13) are also a system of nonlinear equations which may be linearized by expansion of a Taylor series and evaluated about the nominal or estimated trajectory. Define the observation deviation as

$$\delta z(t) = z(t) - z_N(t) \quad . \quad (28)$$

The observations contain only additive noise so that when the Taylor series expansion is performed, as was done previously, the results are

$$\delta z(t) = \left. \frac{\partial h[x_N(t), t]}{\partial x_N(t)} \right|_{x_N} \delta x_N(t) + v(t) \quad . \quad (29)$$

(observation deviation evaluated about the nominal)

Similarly,

$$z(t) = \left. \frac{\partial h[\hat{x}(t), t]}{\partial \hat{x}(t)} \right|_{\hat{x}} x(t) + v(t) \quad (30)$$

(observation evaluated about the estimate)

Before expressing the algorithms for the suboptimal filter the following definitions will be useful:

$$F_N(t) \triangleq \left. \frac{\partial f[x_N(t), w(t), t]}{\partial x_N(t)} \right|_{x_N}, \quad (31)$$

$$G_N(t) \triangleq \left. \frac{\partial f[x_N(t), w(t), t]}{\partial w(t)} \right|_{w_N}, \quad (32)$$

$$H_N(t) \triangleq \left. \frac{\partial h[x_N(t), t]}{\partial x_N(t)} \right|_{x_N}, \quad (33)$$

$$F_E(t) \triangleq \left. \frac{\partial f[\hat{x}(t), w(t), t]}{\partial \hat{x}(t)} \right|_{\hat{x}}, \quad (34)$$

$$G_E(t) \triangleq \left. \frac{\partial f[\hat{x}(t), w(t), t]}{\partial w(t)} \right|_{\hat{w}}, \quad (35)$$

$$H_E(t) \triangleq \left. \frac{\partial h[\hat{x}(t), t]}{\partial \hat{x}(t)} \right|_{\hat{x}}, \quad (36)$$

Using the definitions previously described, the filter equations that describe the state deviations for the two mechanizations follow in Tables I and II. Derivations of these equations can be found in references [1], [2], and [6] through [9].

Table I. Linearized Kalman Filter Equations

Filter state deviation	$\delta \hat{\mathbf{x}}(t) = \mathbf{F}_N(t) \delta \hat{\mathbf{x}}(t) + \mathbf{K}(t) [\delta z(t) - \mathbf{H}_N(t) \delta \hat{\mathbf{x}}(t)]$
Suboptimal gain	$\mathbf{K}(t) = \Sigma(t) \mathbf{H}_N^T(t) \mathbf{R}^{-1}(t)$
Error equation	$\delta \tilde{\mathbf{x}}(t) = \delta \mathbf{x}(t) - \delta \hat{\mathbf{x}}(t)$
Error covariance matrix	$\Sigma(t) = E\{\delta \tilde{\mathbf{x}}(t), \delta \tilde{\mathbf{x}}^T(t)\}$
Error covariance differential equation	$\dot{\Sigma}(t) = \mathbf{F}_N(t) \Sigma(t) + \Sigma(t) \mathbf{F}_N^T(t)$ $- \Sigma(t) \mathbf{H}_N^T(t) \mathbf{R}^{-1}(t) \mathbf{H}_N(t) \Sigma(t)$ $+ \mathbf{G}_N(t) \mathbf{Q}(t) \mathbf{G}_N^T(t)$
Best complete linearized state estimate	$\hat{\mathbf{x}}(t) = \mathbf{x}_N(t) + \delta \hat{\mathbf{x}}(t)$

Table II. Extended Kalman Filter Equations

Filter State equation	$\dot{\hat{x}}(t) = f[\hat{x}(t), t] + K(t) [z(t) - h(\hat{x}(t), t)]$
Suboptimal gain	$K(t) = \Sigma(t) H_E^T(t) R^{-1}(t)$
Error equation	$\tilde{x}(t) = x(t) - \hat{x}(t)$
Error covariance matrix	$\Sigma(t) = E[\tilde{x}(t), \tilde{x}^T(t)]$
Error covariance differential equation	$\begin{aligned} \dot{\Sigma}(t) = & F_E(t) \Sigma(t) + \Sigma(t) F_E^T(t) \\ & - \Sigma(t) H_E^T(t) R^{-1}(t) H_E(t) \Sigma(t) \\ & + G_E(t) Q(t) G_E^T(t) \end{aligned}$

6. Development of the Filter Models

a. The Explicit Linearized Equations

The algorithms summarized in Tables I and II require the evaluation of partial derivatives about the nominal and estimated trajectory. Equation (31) is a matrix operation which is derivable from the additional information which follows. Because

$$f[x(t), w(t), t] = \begin{bmatrix} f_1[x(t), w(t), t] \\ \cdot \\ \cdot \\ \cdot \\ f_n[x(t), w(t), t] \end{bmatrix} \quad (37)$$

and

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} ; \quad (38)$$

then,

$$F_N(t) = \frac{\partial f[x_N(t), w(t), t]}{\partial x_N(t)} \bigg|_{x_N} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \frac{\partial f_n}{\partial x_1} & \cdot & \cdot & \cdot & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x_N}, \quad (39)$$

where the arguments have been dropped for simplicity. It is obvious, but worth repeating, that

$$f_1[x(t), w(t), t] = \dot{h} = v \sin \gamma, \quad (40)$$

$$f_2[x(t), w(t), t] = \dot{d} = \frac{R_e v \cos \gamma}{h + R_e}, \quad (41)$$

$$f_3[x(t), w(t), t] = \dot{v} = \frac{-k \sin \gamma}{(h + R_e)^2} + g \left(\frac{T}{w_o} \right) \frac{1}{1 - \left(\frac{T}{w_o} \right) \frac{t}{I_{sp}}}, \quad (42)$$

$$f_4[x(t), w(t), t] = \dot{\gamma} = \frac{d}{R_e} - \frac{k \cos \gamma}{v(h + R_e)^2} + \frac{F_n}{v}, \quad (43)$$

and that

$$x_1(t) \triangleq h(t) \quad , \quad (44)$$

$$x_2(t) \triangleq d(t) \quad , \quad (45)$$

$$x_3(t) \triangleq v(t) \quad , \quad (46)$$

$$x_4(t) \triangleq \gamma(t) \quad . \quad (47)$$

By performing the differentiation described by Equation (39), the following four by four array is obtained:

$$F_N = \begin{bmatrix} 0 & 0 \sin \gamma & v \cos \gamma \\ \frac{-R_e v \cos \gamma}{(h + R_e)^2} & 0 \frac{R_e \cos \gamma}{h + R_e} & \frac{-R_e v \sin \gamma}{h + R_e} \\ \frac{2k \sin \gamma}{(h + R_e)^3} & 0 & 0 & \frac{-k \cos \gamma}{(h + R_e)^2} \\ \frac{[-v^2(h + R_e) + 2k] \cos \gamma}{v(h + R_e)^3} & 0 & \frac{[v^2(h + R_e)^2 + k] \cos \gamma}{v^2(h + R_e)^2} & \frac{[-v^2(h + R_e)^2 + k] \sin \gamma}{v(h + R_e)^2} \end{bmatrix} . \quad (48)$$

By an analogous operation described by Equation (32), the following array for $G_N(t)$ is completely filled with zeros except for the element in the fourth row and fourth column:

$$G_N(t) = \begin{bmatrix} & & & 0 \\ & \bigcirc & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix} w_N \quad . \quad (49)$$

For the observation equations, the functions are

$$h_1[x_N(t), t] = S = \sqrt{r^2 + R_e^2 - 2R_e r \cos(\theta_R - \theta)} \quad , \quad (50)$$

and

$$h_2[x_N(t), t] = L = \arctan \left(\frac{\cos(\theta_R - \theta) - \frac{R_e}{r}}{\sin(\theta_R - \theta)} \right) \quad . \quad (51)$$

Performing the operation described by Equation (33) results in the following two by four array:

$$H_N(t) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial x_4} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_4} \end{bmatrix}_{x_n} \quad . \quad (52)$$

The arguments on the partials have been left out for simplicity and

$$\frac{\partial h_1}{\partial x_3} = \frac{\partial h_1}{\partial x_4} = \frac{\partial h_2}{\partial x_3} = \frac{\partial h_2}{\partial x_4} = 0 \quad ; \quad (53)$$

also,

$$\frac{\partial h_1}{\partial x_1} = \frac{R_e}{s} \left(1 + \frac{h}{R_e} - \cos(\theta_R - \theta) \right) \quad , \quad (54)$$

$$\frac{\partial h_1}{\partial x_2} = - \frac{R_e}{s} \left(1 + \frac{h}{R_e} \right) \sin(\theta_R - \theta) \quad , \quad (55)$$

$$\frac{\partial h_2}{\partial x_1} = \frac{\sin(\theta_R - \theta)}{R_e \left(1 + \frac{h}{R_e}\right)^2 \left(1 - \frac{2 \cos(\theta_R - \theta)}{1 + \frac{h}{R_e}}\right) + \frac{1}{\left(1 + \frac{h}{R_e}\right)^2}}, \quad (56)$$

and

$$\frac{\partial h_2}{\partial x_2} = \frac{\left(1 - \frac{\cos(\theta_R - \theta)}{1 + \frac{h}{R_e}}\right)}{R_e \left(1 - \frac{2 \cos(\theta_R - \theta)}{\left(1 + \frac{h}{R_e}\right)} + \frac{1}{\left(1 + \frac{h}{R_e}\right)^2}\right)}. \quad (57)$$

The arrays for the expressions given by Equations (34), (35), and (36) are exactly the same with the important restriction that they are to be evaluated about the current estimate and not about the nominal.

b. Computer Synthesis of Suboptimal Estimators

The synthesis of a computer algorithm for the computations required to estimate the four states of the idealized rocket launch can be observed in Figures 2 through 5. These are similar to results derived by Lange [10]. Figure 2 is a block diagram of the overall program with the linearization done about a nominal trajectory. Figure 3, in more detail, describes the linearized filter process. Figure 4, in more detail, describes the actual and nominal trajectories.

The synthesis for the extended filter is essentially the same but with the major exception that the estimated state equations replace the nominal state equations in Figures 2 and 4. The estimated equations are formed by adding the observation differences multiplied by the optimal gain. Figure 5 shows this process.

c. Discussion of the Model and Problem

The problem, mentioned earlier, is to estimate four parameters of an idealized vehicle's trajectory launched from a spherical, airless, nonrotating earth. The trajectory parameters are altitude (h), downrange distance traveled as measured from the launch site (d), velocity (v), and flight path angle (γ). The launch site and trajectory of the vehicle are in a plane which contains a radar tracking station

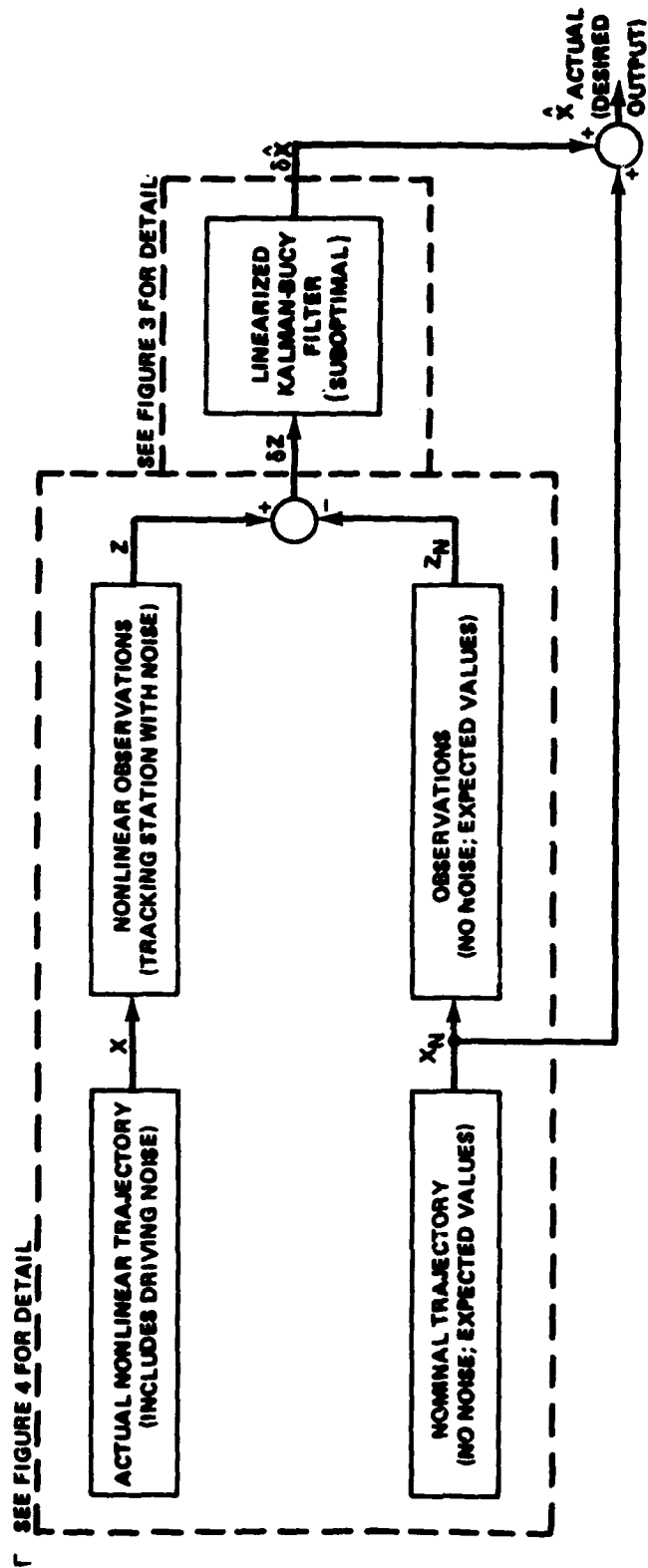


Figure 2. Overall Synthesis of Linearized Filter

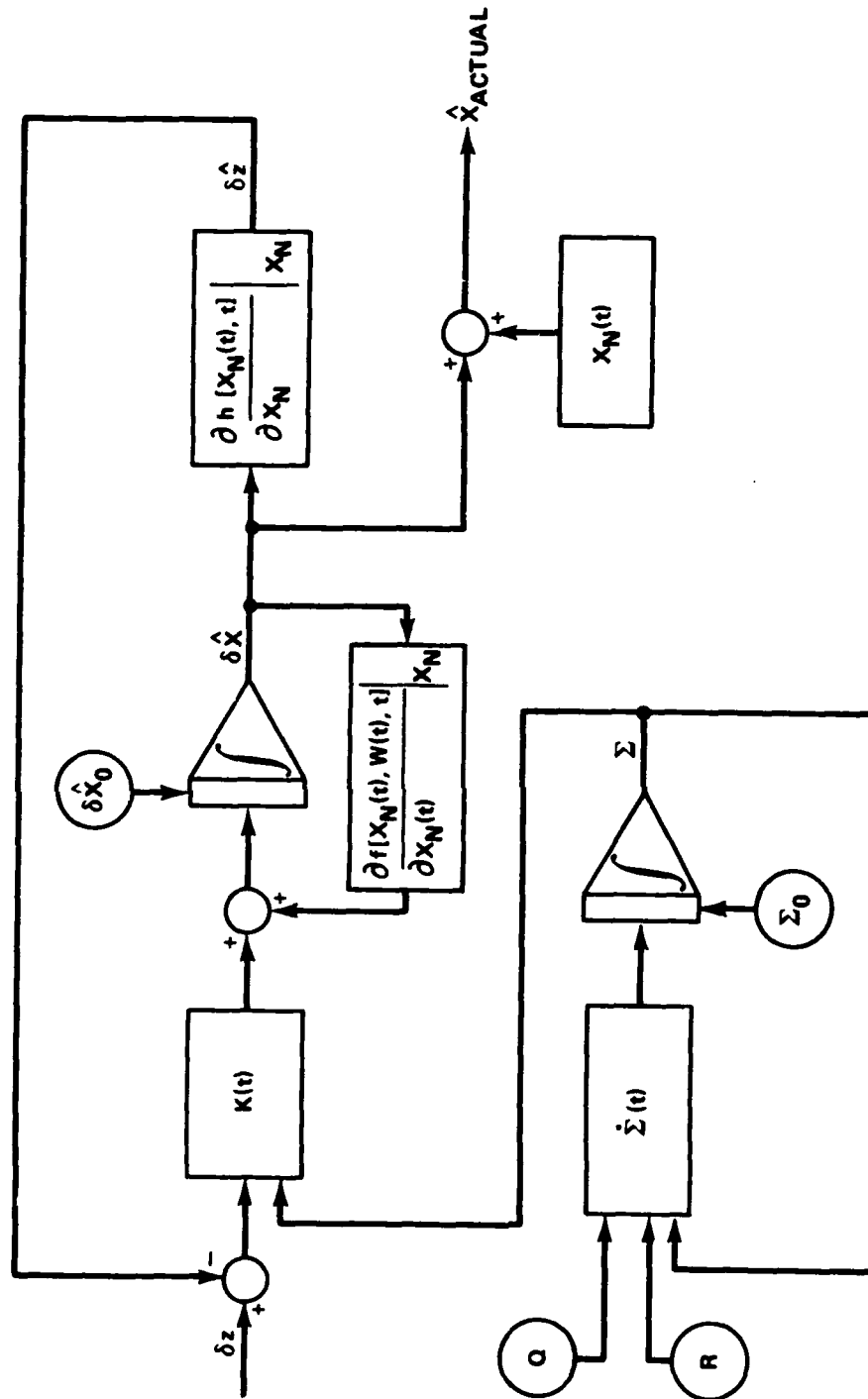


Figure 3. Linearized Filter Detail

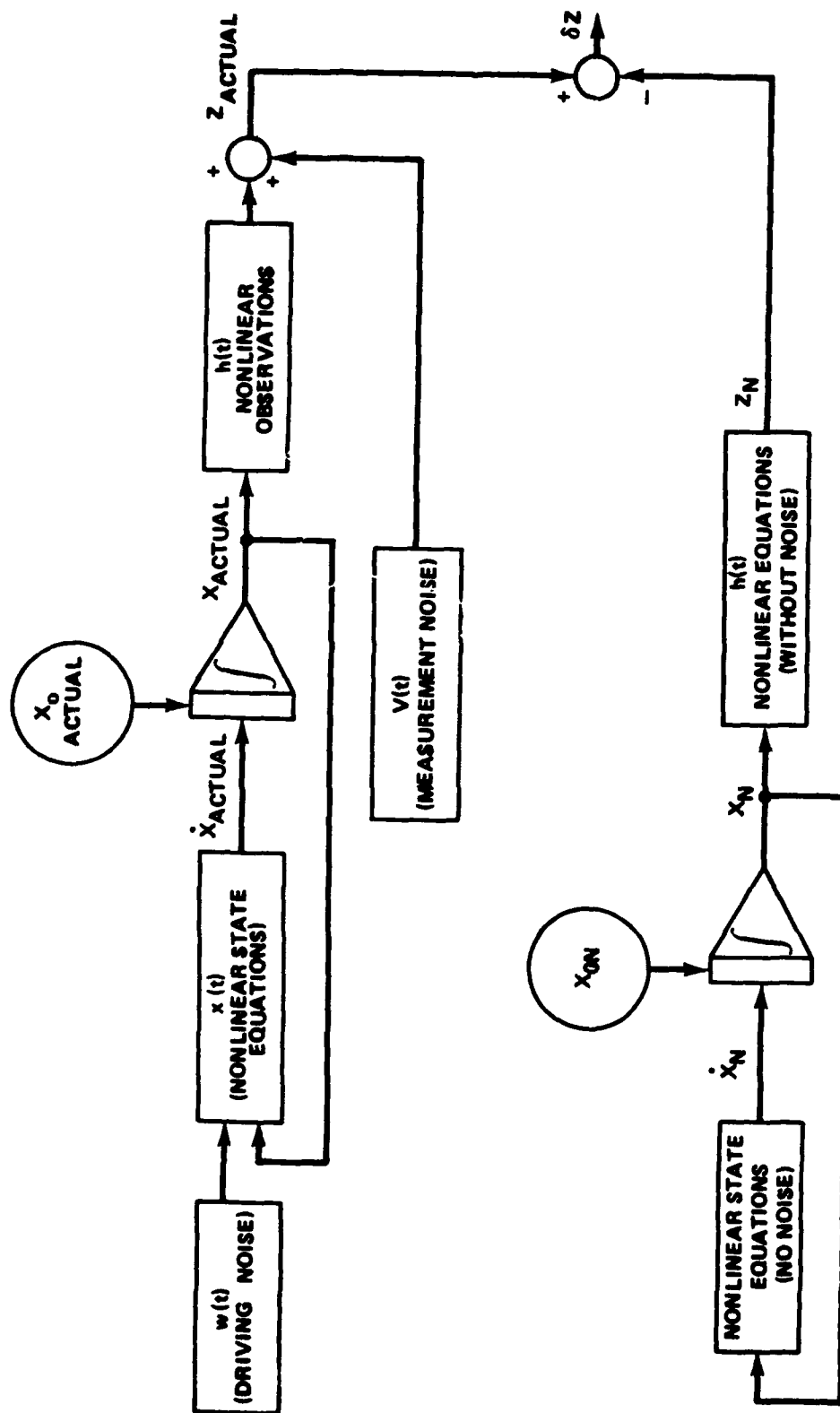


Figure 4. Detail of Nominal and Actual Trajectory Development

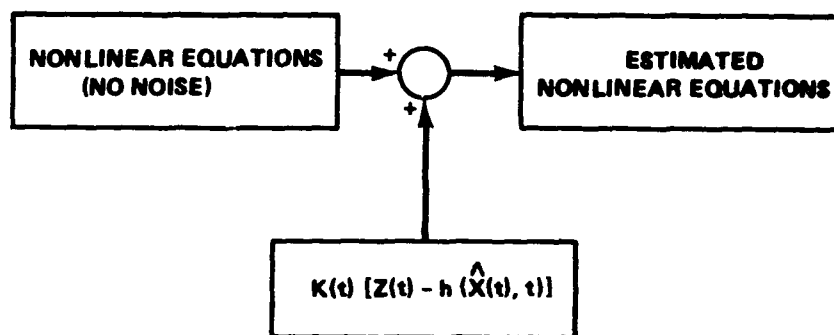


Figure 5. Formation of Estimated Nonlinear Equations

located 40 nautical miles from the launch site. The simulation is run with the noise (w), which is considered a disturbing wind, at a value of zero until 15 seconds has elapsed. This can be envisioned as a case where the surface winds are negligible for altitudes up to about 2500 feet.

The trajectory is planned to achieve an altitude of about 80 miles with a flight path angle suitable for injection into a circular orbit. Thus, at the time of 220 seconds, the ideal case would be to have γ equal to zero.

In the runs where the Extended Kalman Filter is used, the initial estimated equations are assumed to be the same as the actual equations but without noise. After 10 seconds the estimated equations are generated more accurately by employing the observation modified by the optimal gains as summarized in Table II.

Also, the elevation angle (L) measured by the tracking radar is not computed for approximately 10 seconds. This is to allow the vehicle, which initially appears below the horizon to the radar tracker, to become observable. The impact of this modification is to prevent the measurement noise covariance matrix (R) from being computed while it is zero. This keeps the computer from reaching an exponential overflow, i.e., $(R)^{-1}$ becomes infinite and the simulation terminates prematurely.

In the initialization of the problem, several constants, initial conditions, and standard deviations of measurement error had to be chosen. Tables III and IV are compilation of those values.

Table III. State Initial Conditions and Constants

States	Constants
$v_o = 100 \text{ ft/sec}$	$T/w_o = 1.3$
$d_o = 0 \text{ ft}$	$R_e = 20.89 \times 10^6 \text{ ft}$
$h_o = 0 \text{ ft}$	$I_{sp} = 300 \text{ sec}$
$\gamma_o = 89.66 \text{ deg}$	$g = 32.17 \text{ ft/sec}^2$
	$d = 40 \text{ n mi}$

NOTE: Because this analysis assures Gaussian statistics, the standard deviation about a zero mean defines the probability distribution function precisely. The 1- σ values for the driving noise and measurement noise are given in Table IV.

Table IV. Standard Deviation of Noise

Driving Noise	Measurement Noise
$\sigma_v = 0 \text{ ft/sec}$	$\sigma_s = 100 \text{ ft}$
$\sigma_d = 10 \text{ ft}$	$\sigma_L = 0.01 \text{ rad}$
$\sigma_h = 100 \text{ ft}$	
$\sigma_\gamma = 0.001 \text{ deg}$	

Finally, the covariance matrices (Σ) , (R) , and (Q) are initialized with the appropriate element given by the square of the standard deviations of Table IV. The initial array for the error covariance matrix is

$$\Sigma_o = \begin{bmatrix} 10^4 & 0 & 0 & 0 \\ 0 & 10^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^6/(57.7)^2 \end{bmatrix}, \quad (58)$$

the initial array for the measurement noise covariance matrix is

$$Q_o = \begin{bmatrix} & & & | & 0 \\ & \bigcirc & & | & 0 \\ & & & | & 0 \\ \hline 0 & 0 & 0 & | & 10^4 g^2 \end{bmatrix} , \quad (59)$$

and the initial array for the inverse of the process noise covariance matrix is

$$R_o^{-1} = \begin{bmatrix} 10^{-4} & 0 \\ 0 & 10^4 \end{bmatrix} . \quad (60)$$

7. Results and Conclusions

The performance of the two mechanizations used to estimate the trajectory states are observable in Figures 6 through 18. The values of plus and minus one standard deviation from the indicated covariance matrix are plotted in Figures 6 through 13. That is, each figure has the square root of its corresponding diagonal element of the covariance matrix plotted for a positive and negative 1- σ value. Also, each figure shows the error of the filter in estimating that state. For the Extended Kalman Filter Mechanization (Figures 6 through 9), the error is given as the difference between the actual state and the estimate of the actual state. For the Linearized Kalman Filter Mechanization (Figures 10 through 13), the filter error is given as the difference between the actual state and the best estimate of the actual state which in turn has been differenced about a nominal, i.e.,

$$\tilde{x} = x_{\text{actual}} - \hat{x}_{\text{actual}} = x_{\text{actual}} - (x_{\text{nominal}} - \delta \hat{x}) . \quad (61)$$

It is interesting to note that the results of both mechanizations indicate proper performance of the filter. This is concluded from the figures by observing that the irregular or "noise-like" trace is indeed within the $\pm 1 \sigma$ curves about 63 percent of the time as the linearized theory suggests.

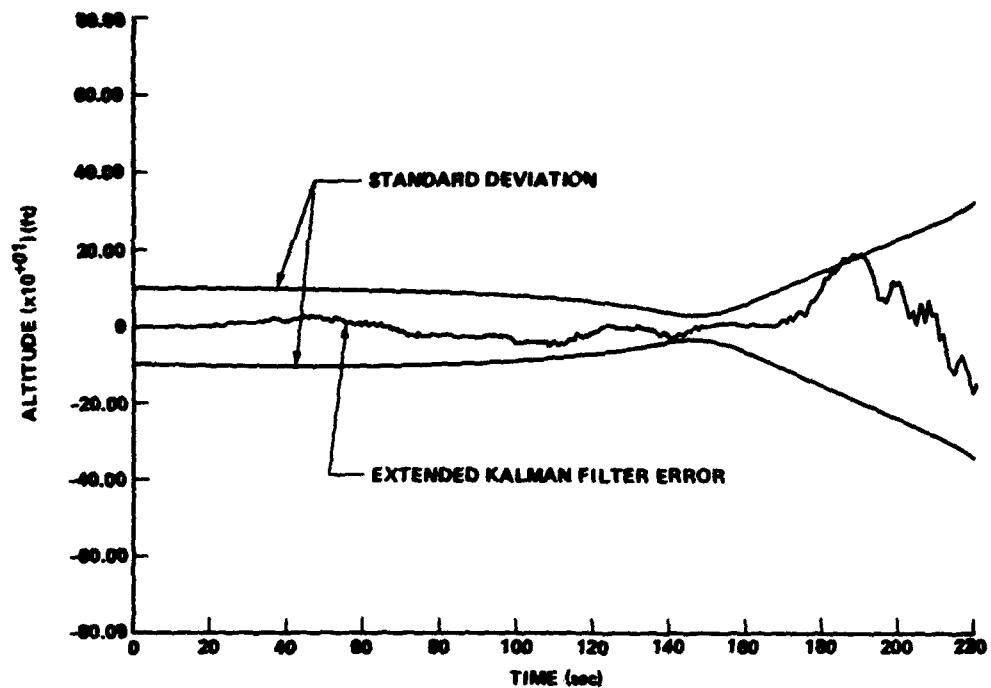


Figure 6. Performance of the Extended Kalman Filter

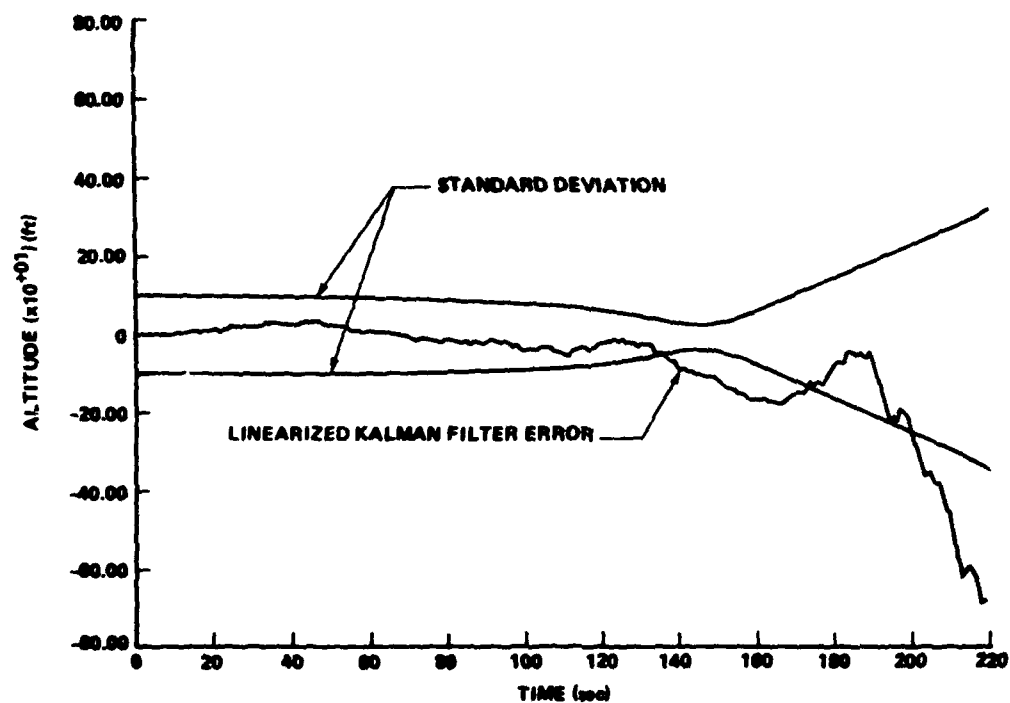


Figure 7. Performance of the Linearized Kalman Filter

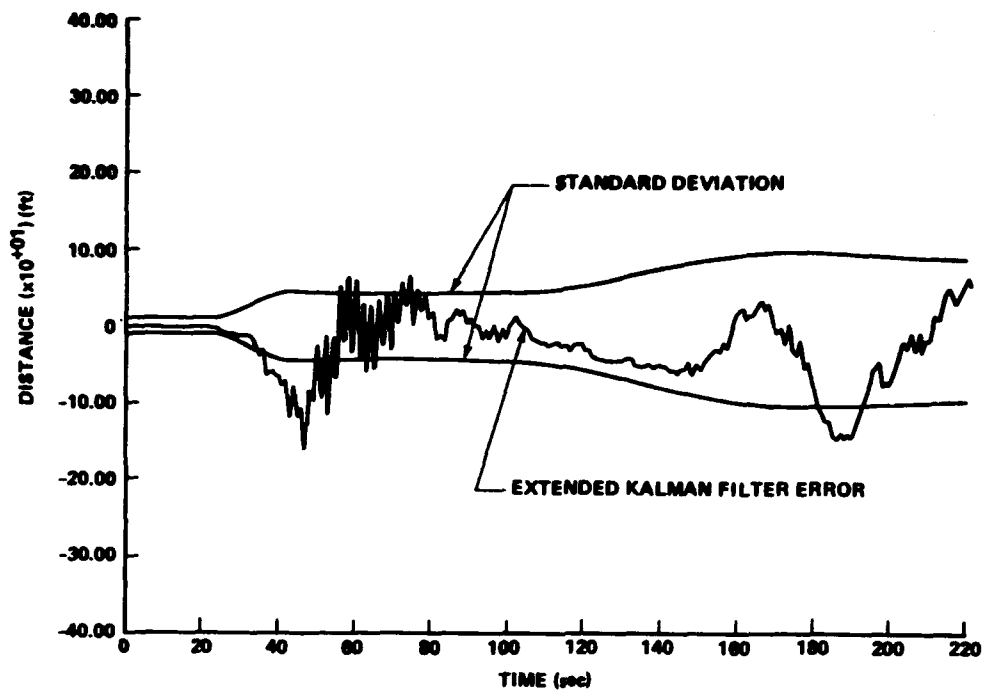


Figure 8. Performance of the Extended Kalman Filter

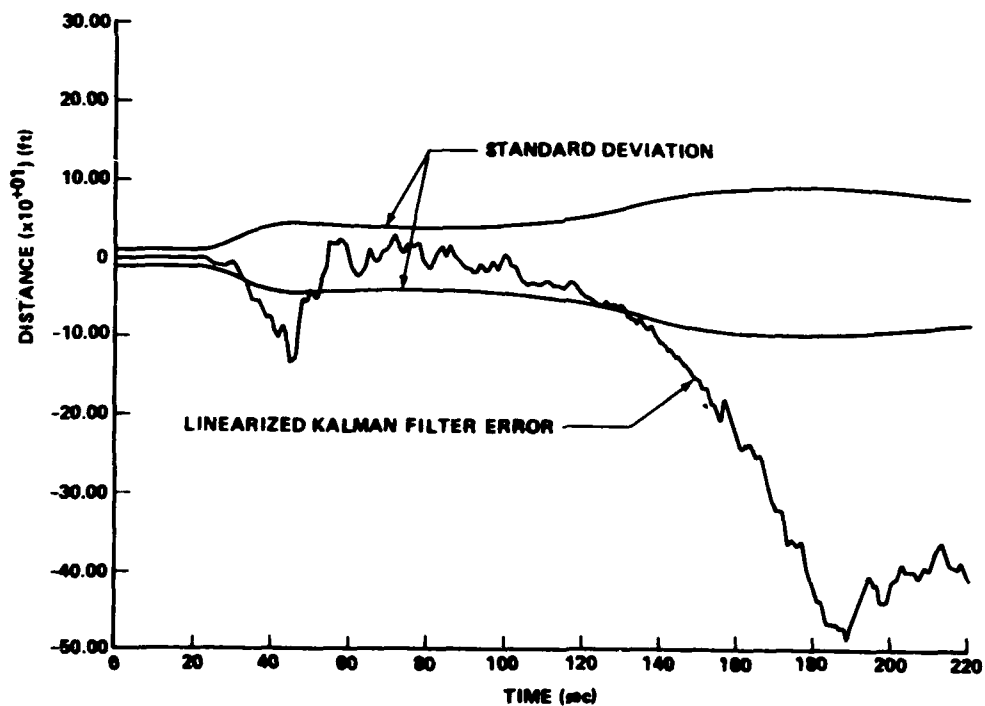


Figure 9. Performance of the Linearized Kalman Filter

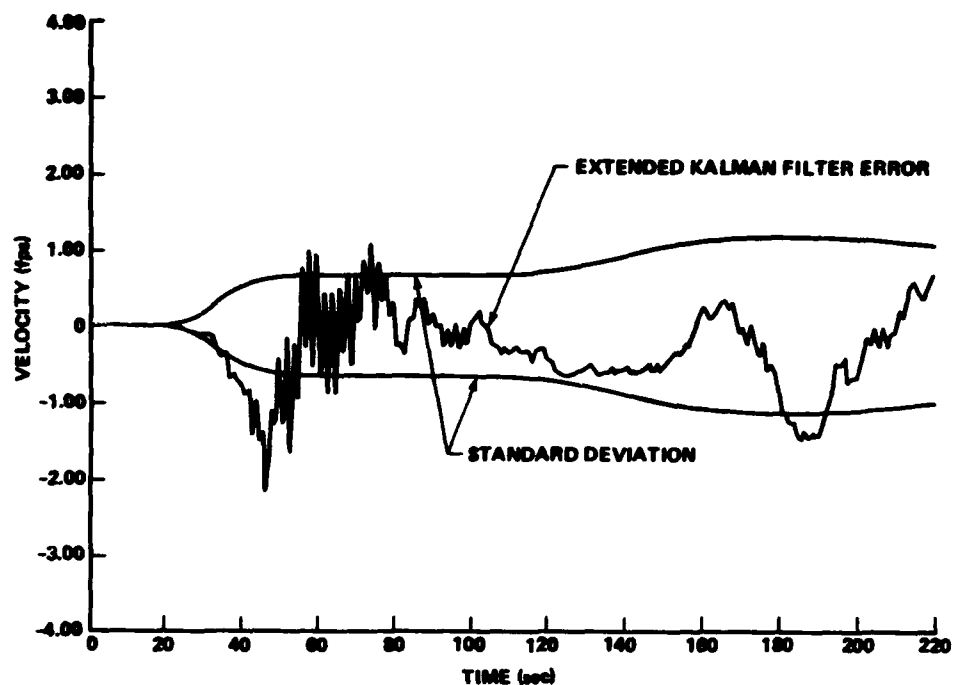


Figure 10. Performance of the Extended Kalman Filter

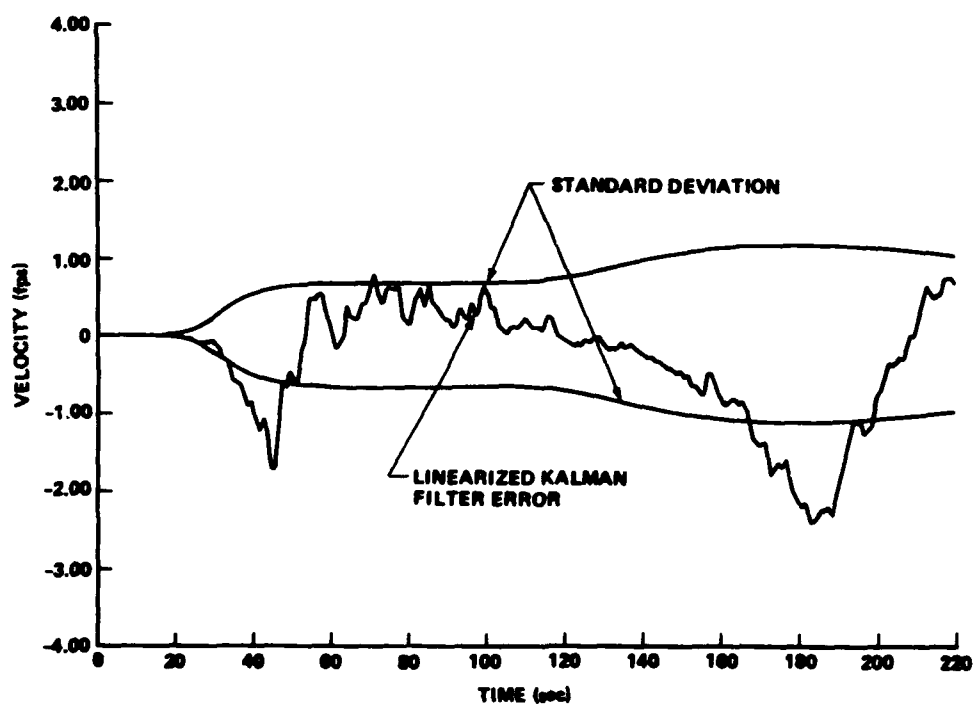


Figure 11. Performance of the Linearized Kalman Filter

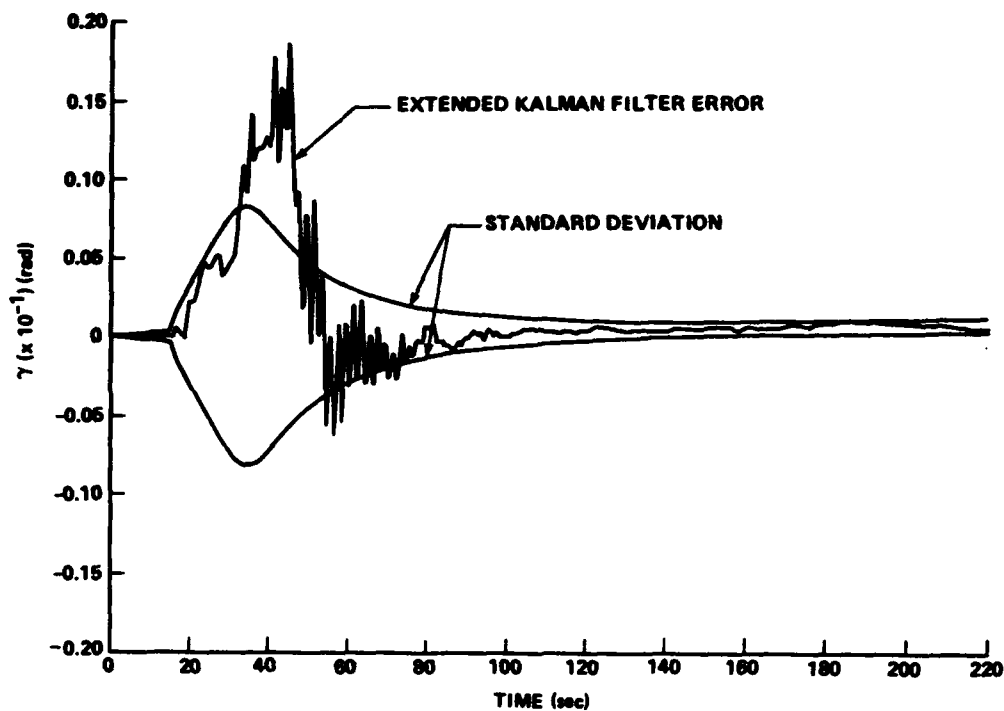


Figure 12. Performance of the Extended Kalman Filter

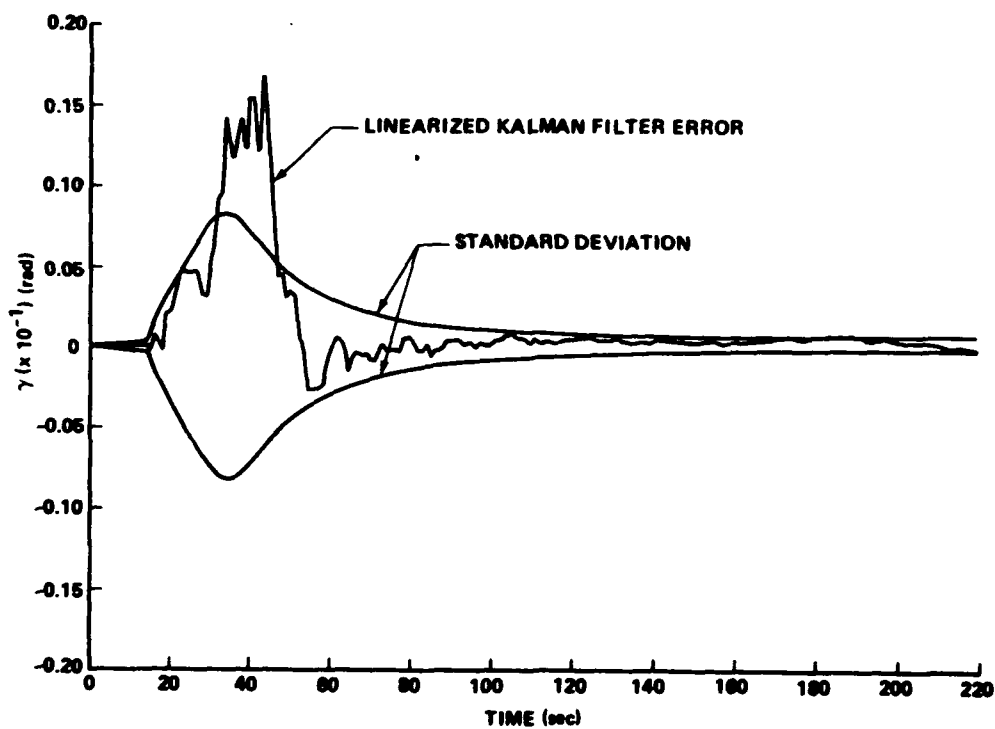


Figure 13. Performance of the Linearized Kalman Filter

On an overlay depicting the extended filter variance versus the linearized filter variance, there is virtually no discernible difference between the two for any of the states estimated. However, Figures 14 through 18 show a graphic view of the filter error in estimating the states. The velocity (v), distance downrange (d), and altitude (h) are estimated with less error with the Extended Kalman Filter Mechanization. The same conclusion holds true for the flight path angle (γ), though not as graphically evident as for the first three states.

At this point one is inclined to conclude that the Extended Kalman Filter (known also as the Relinearized Kalman Filter) is superior to the Linearized Kalman Filter in estimating states defined by nonlinear equations. However, in reporting on a larger sample of results by others, Jazwinski [9] notes that such results cannot be assessed a priori. That is, though better more often than not, the Extend Filter can sometimes be worse and may even diverge (Kushner [3]).

The future effort in the application of these mechanizations to nonlinear filtering problems is probably best directed to actual simulations as was done here. More experience with the approximate nonlinear filters is desirable because the insight gained in this type of work may lead to other useful approximation techniques.

8. Future Applications

This report describes the first phase of a study directed toward application in the Army's Cannon Launched Guided Projectile (CLGP). It is well known that the Proportional Navigation and Guidance Law used in the CLGP, though highly accurate for its mission, is susceptible to the effects of the gravity field force. The result is that the projectile's flight trajectory has a tendency to "droop" or fall short of the intended target at low quadrant elevation firings. One way to minimize this effect is to roll stabilize the projectile and, knowing the direction of the vertical, bias the autopilot to, in essence, aim high and counteract gravity.

A common method for roll stabilization is to measure roll rate with a gyroscope sensor and null the vehicle's roll rate to some small level. In the CLGP environment, however, the gyroscopic sensor is subjected to extremely high acceleration forces at the launching. Thus, the question may be asked if an alternative method of measuring and nulling roll rate is available. The answer leads into the program for which this report was initiated. The program is to determine the feasibility of estimating roll rate of the CLGP vehicle without the requirement of a specific additional roll rate sensor. The solution, if it is feasible, would make use of other on-board sensors, such as the laser seeker, to estimate the required state. Implementation of this scheme is through formulation of a Kalman-Bucy filter which, in the linear theory, optimally yields the best estimate in a minimum variance sense.

Because the CLGP equations of motion are not linear, the problem is complicated by the choice of a linearization scheme. This report addresses that problem, in particular, for a simplified and idealized system of equations. However the results are directly applicable to the future work described above.

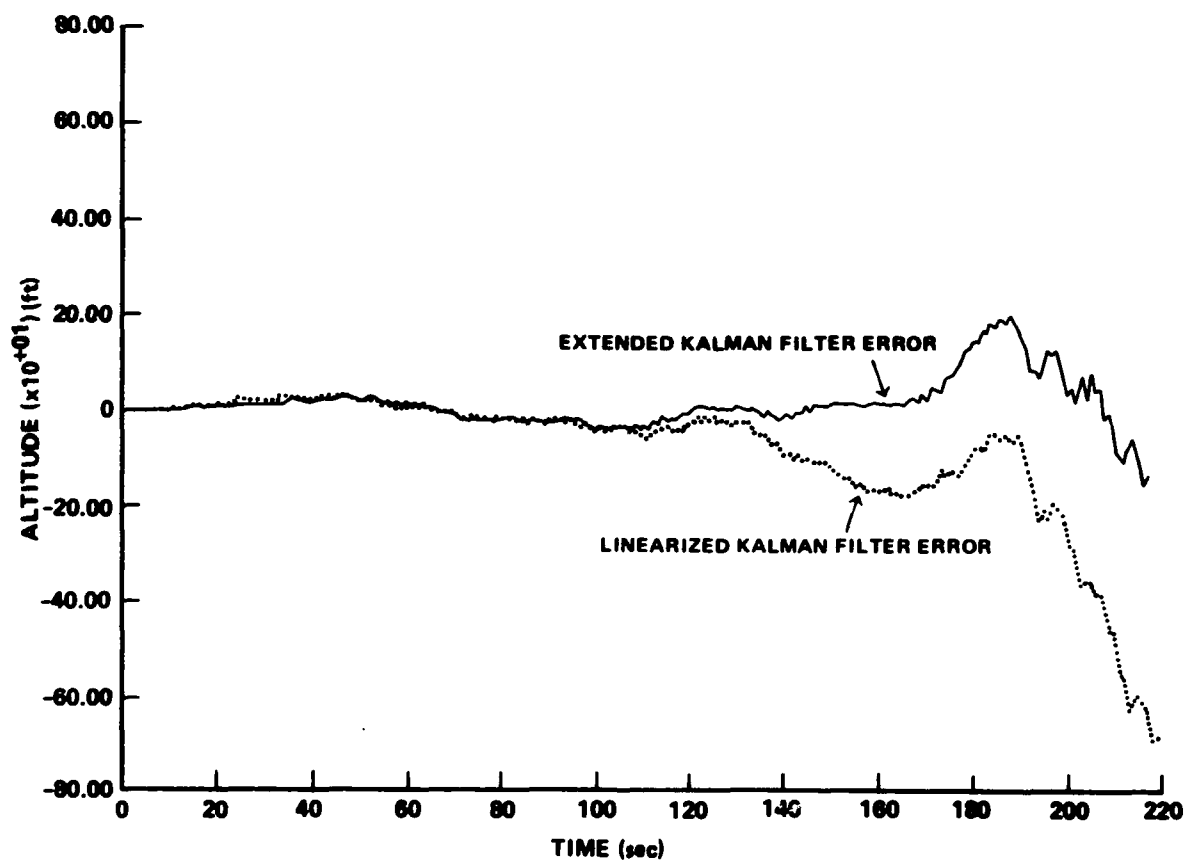


Figure 14. Comparison of Filter Errors

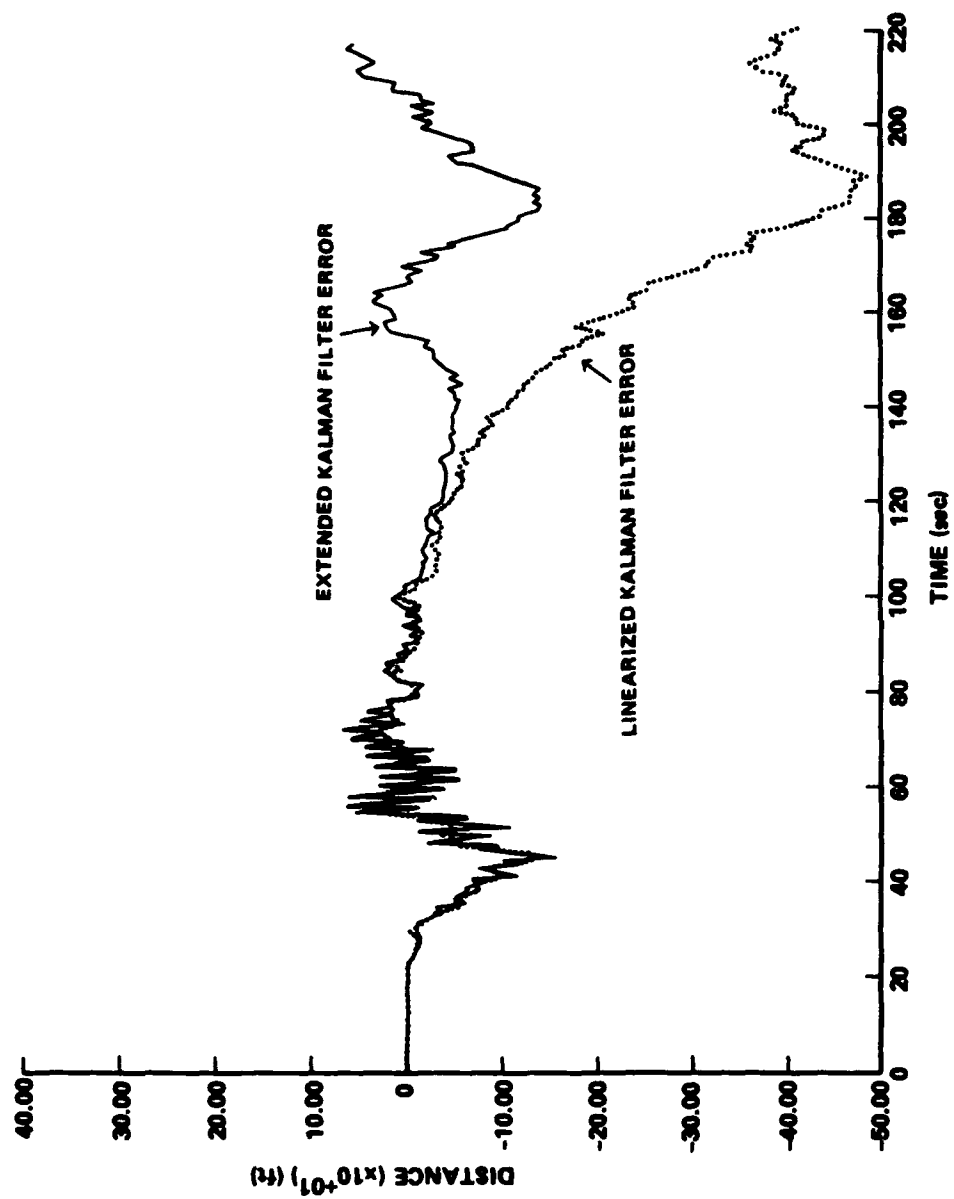


Figure 15. Comparison of Filter Errors

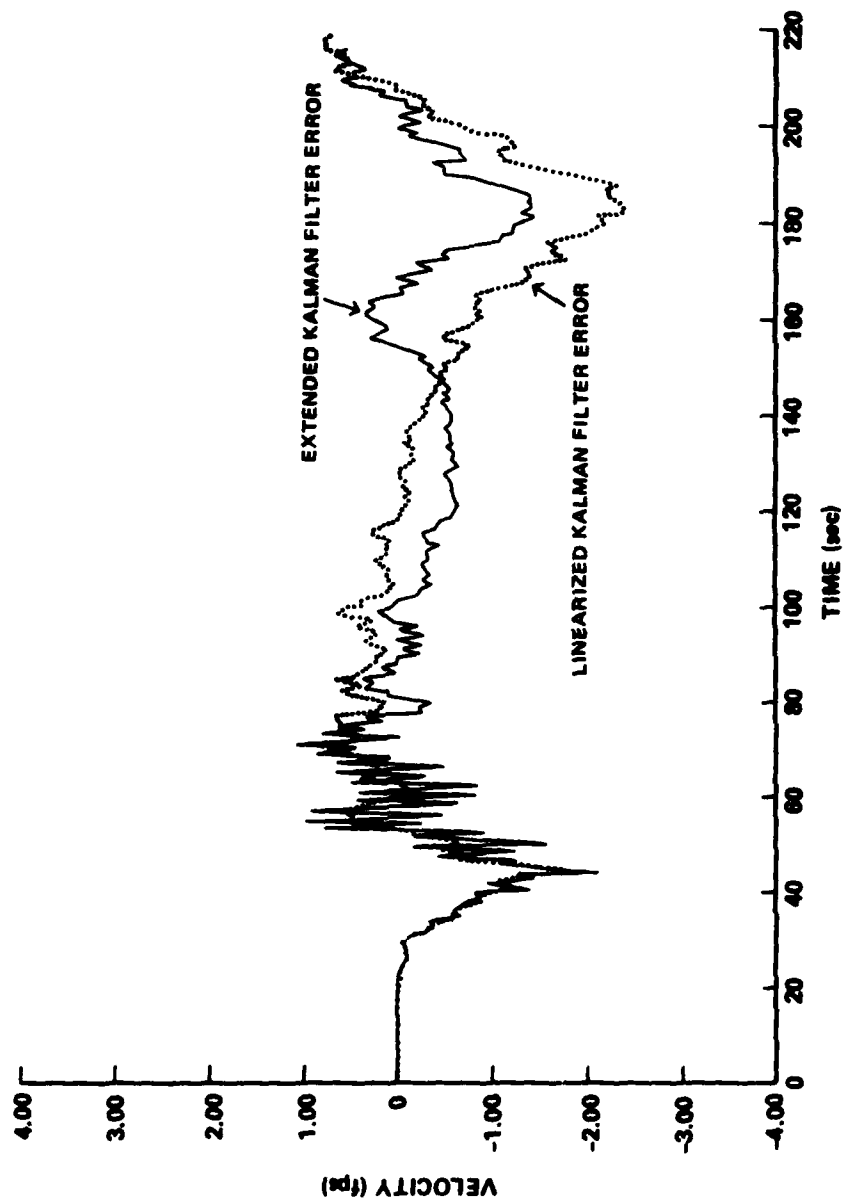


Figure 16. Comparison of Filter Errors

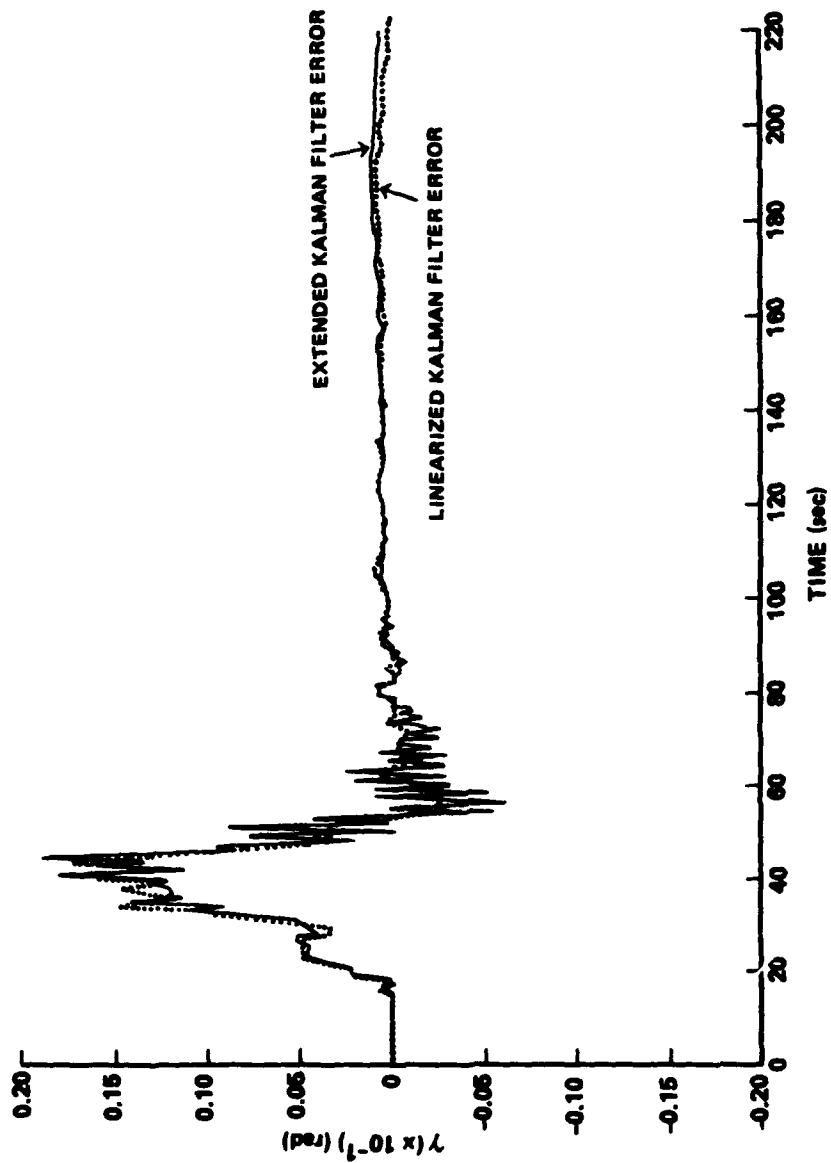


Figure 17. Comparison of Filter Errors

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